Uncertainties in Modeling Low Probability/High Consequence Events: Application to Population Projections and Models of Sea-level Rise

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Abstract

We present a simple method for estimating uncertainty in modeling and forecasts based upon an analysis of errors in old measurements and projections. Probabilities of large deviations are parametrized by an exponential function with one free parameter. We illustrate this formulation by quantifying uncertainties in national population projections and by estimating the probability of extreme sea-level rise resulting from global warming.

1 Introduction

Science policy frequently hinges on reliable assessment of the uncertainties in predictions derived from model based forecasting [1,2]. Our analysis of several datasets of physical measurements, U.S. energy demand, and population projections [3-6], shows that there is a general pattern of overconfidence in empirical probability distributions. The errors are manifest in the normalized deviations of the new values from the old ones. Although the exact causes of such errors are specific to each discipline, the pattern of overconfidence in measurements and model results are common. We find that the probability of large deviations from the expected means can be conveniently parametrized by exponential functions. In this paper we build on earlier work [1-7] and develop an empirical method of quantifying the uncertainty in a time-series of historical forecasts for which the actual values are now known. The goal here is to illustrate the method by applying revised uncertainty estimates to current forecasts.

2 Distribution of Uncertainties in Physical Measurements

A convenient measure of the deviation of "old" values from the "true" values is $x = (a - A)/\Delta$, with a the exact value, A the previously measured value, and Δ the old standard error. For physical measurements, it is usually assumed that x values follow normal distributions. We recently analyzed several datasets of physical measurements [3-6]. The results indicate that large deviations from the previously estimated values occur much more often than is predicted by the Gaussian distribution. This is illustrated in Figure 1 where the "empirical" cumulative probability distributions of $|x|$ for five datasets derived from nuclear physics are shown together with the cumulative Gaussian curve, which obviously grossly underestimates probability of large deviations (see [5] for details regarding these data). The discrepancy between the Gaussian distribution of uncertainties and distributions observed experimentally cannot be removed simply by inflating the normal distribution by increasing its variance. Even an inflated Gaussian distribution is a poor fit to the data: as seen in Figure 1 the data follow exponential distribution which is a straight line in semi-log plots while Gaussian curves are parabolic.

Bukhvostov [8] and Shlyakhter and Kammen [3,4] put forward simple arguments to justify an exponential parametrization. Let us assume that the mean, *A*, is unbiased but that the estimate of the true standard deviation, Δ', is randomly biased by systematic errors with a distribution *f(t)* where $t=\Delta'/\Delta$. Here Δ is the estimated standard deviation. In other words we assume that $x'=(a-A)/\Delta'$ follows the standard normal distribution while *x* values follow a normal distribution with standard deviation *t*:

$$p_t(x)=\frac{1}{\sqrt{2\pi}\,t}e^{-\frac{x^2}{2t^2}} \quad (1)$$

Integrating over all values of *t* gives a compound distribution:

$$p(x)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{dt}{t}f(t)\,e^{-\frac{x^2}{2t^2}} \quad (2)$$

If f(t) has a sharp peak near t=1, Eq. (2) reduces to the normal distribution. If f(t) is broad, however, the result is different. For simplicity, we shall consider only the asymptotic behavior of p(x) when $|x|>>1$. In this case, the value of the integral in Eq.(2) is determined by the asymptotic behavior of f(t) as $t\to\infty$, since for small t the exponent is nearly zero. Let us assume that for t >> u f(t) follows a Gaussian law:

$$f(t)\propto e^{-\frac{t^2}{2u^2}} \quad (3)$$

where $u=\delta/\Delta$. The new parameter u, comprises the unknown systematic component of the total error and quantifies the uncertainty δ in the estimation of Δ. At $|x|>>1$ the main contribution to the integral in Eq.(1) comes from the vicinity of the saddle point where the exponential term reaches a maximum (for $t=t_{max}$: $t^2_{max}=u|x|$). For $|x|>1$, the probability distribution is not *Gaussian* but *exponential*:

$$p(x)\propto e^{-\frac{|x|}{u}} \quad (4)$$

In this paper we use a truncated normal distribution for *f(t);* where *f(t)* is zero for t<1 and follows Gaussian distribution with the mean value *t*=1 and standard deviation u for *t*>1 (multiplied by a factor of two to maintain the proper normalization of the probability density). Formally, we have:

$$f(t)=0,\ t\le 1$$
$$f(t)=\sqrt{\frac{2}{\pi}}\,\frac{1}{u}\,e^{-\frac{(t-1)^2}{2u^2}},\ t>1 \quad (5)$$

This choice of *f(t)* is consistent with Eq. (2) and reflects the fact that *t*<1 is highly improbable as it corresponds to *under-confidence* (estimated standard deviation $\Delta' < \Delta$) and negative values of *t* that are physically impossible. Note that in [4] we used slightly different form of f(t) (truncated at *t*=0 instead of *t*=1). A family of *f(t)* for different u values is shown in Figure 2. The parametrization chosen here has the advantage that the effect of truncation does not depend on the value of *u*. Integrating Eq. (2) gives the cumulative probability *S(x)* of deviations exceeding $|x|$:

$$S(x)=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{u}\int_1^{\infty}e^{-\frac{(t-1)^2}{2u^2}}\,\mathrm{erfc}\left(\frac{|x|}{t\sqrt{2}}\right)dt \quad (6)$$

For *u*=0 Eq. (6) is reduced to $S(x)=erfc(|x|/\sqrt{2})$ and the probability distribution is Gaussian. On a logarithmic scale this curve is parabolic while the exponential curves ($|x| > 0$) are linear. As shown in Figure 2, at $u \sim 1$ Gaussian distribution underestimates the probability of extreme events ($x \geq 5$) by several orders of magnitude.

The normal ($u = 0$) and exponential ($u >$ distributions 1) are members of a single-parameter family of curves (see Figure 2). For quick estimates for $u \geq 1$, $x \geq 3$, one can use the approximation $e^{-|x|/(0.7u + 0.6)}$. In this framework the parametric uncertainty can be quantified by analyzing the record of prior projections and estimating the value of u. Our previous analysis shows that $u \sim 1$ for physical constants and $u \sim 3$ for current models of population growth and forecasts of the U.S. energy supply and demand [3-6]. Thus, while u is not the same for different types of forecasts, these data exhibit a consistent functional form that can be easily computed from a set of past projections and subsequent measurements of the true values.

3 Parametrization of the Distribution of Uncertainties in the Forecasts

Uncertainty in future forecasts is defined less formally than uncertainty in physical measurements. In this section we develop an algorithm for analysis of the distribution of x values derived from historical forecasts. We estimate the standard deviation Δ of an equivalent normal distribution and then draw the empirical probability distributions of the deviations of the old forecasts from the true values normalized by Δ. A comparison of the empirical frequency of large deviations from the predicted values with the normal distribution allows an analogy with the well-understood case of stochastic uncertainties.

Uncertainty in the forecasts is usually presented in the form of "reference," "lower" and "upper" estimates (R, L, and U respectively) that are obtained by running a model with different sets of exogenous parameters (e.g. the annual rate of growth or the size of a carbon emissions tax). The range of scatter around the reference value R does not formally define a Gaussian standard deviation because the fundamental uncertainties involved (e.g. the rate of future economic growth) are frequently not stochastic. However, it is reasonable to assume that the range of parameter variation presented by a forecaster represents a subjective judgment [1,2,9] about the probability that the true value $T \in [L, U]$. Generally, lower and upper bounds present what is believed to be an "envelope" most likely to bracket the true value and include the majority of possible outcomes.

The standard deviation of the equivalent normal distribution is calculated as follows:

a) Specify the subjective probability α that the true value will lie between the low (L) and high (U) estimates. We assumed α=68%; larger values of α increase the discrepancy between the Gaussian model and that calculated by this method.

b) Draw an equivalent normal distribution that would have a specified cumulative probability α between L and U. For α = 68% the standard deviation of the equivalent normal distribution is (U - L)/2. Therefore this choice of α corresponds to the usual practice of splitting the uncertainty range in half and using it as a surrogate of standard deviation.

c) If the reference value (R) is not in the middle of the (L, U) interval use two separate normal distributions truncated at zero: "left half" for (L, R) interval and "right half" for (R, U) interval each having $\alpha/2$ as the cumulative probability.

The new (inflated) confidence intervals are calculated by estimating u from the historical data and calculating the new low (LN) and new high (UN) limits as follows:

$$LN = R - Z(R - L)$$

$$UN = R + Z(U - R)$$

where the inflation factor Z can be read from the curves in Figure 2. For the α confidence interval, Z is the ratio of $|x(u)|/|x(u=0)|_{1-\alpha}$. For α=0.95 this gives: Z = 1.9; 3.0; 4.1; and 5.2 for u = 1, 2, 3, 4 respectively.

Note that in estimating u values we assumed α=68% for the *old* forecasts but for the *current* projections we assume α=95%. In this way we account for the (hopefully) improved reliability of more recent forecasts.

Had we assumed α=95% for the old forecasts, the derived standard deviations would be two times smaller and all x values would be two times larger. The resulting u values and the corresponding inflation factors would be also larger than the ones we used.

4 Analysis of Population Projections

The data base of population projections provides an opportunity both to test our method by calculating the cumulative probability distribution for past estimates and to compare the results to existing census figures. From this analysis we hope to develop a picture of the uncertainty in the projections of future population growth. We analyzed United Nations population projections for the year 1985 that can serve as the set of "exact" values, *a*, made in 1972 [10]. The population data base includes projections from 164 nations with population exceeding 100,000 presented in the form of "high" and "medium" and "low" variants for each nation. Data for 31 countries was excluded due to extreme errors (up to $|x|$ = 300) that resulted from unanticipated international migration (frequently war refugees between relatively small nations), reliability questions surrounding particular census efforts, and clear cases of politically motivated reporting bias. Data for 133 nations satisfying the criteria $|x| < 10$ are included in the present study.

Because all the population estimates come from an authoritative source, namely the United Nations, it might be expected that systematic errors would be small, representing a well-calibrated model. The uncertainty, however is very large, characterized by u>3. Data for 37 industrialized countries where data are generally more reliable (see Figure 3) show slightly less surprise and are well described by exponential with u~3.

Our method can be applied to current population projections by inflating the estimated uncertainty range by a numerical factor of about four. This results in revised 95% confidence intervals. For example, the UN projections [10] for population of France in the year 2005 give the lower estimate L=57.96 million, reference value R=58.86 million, and the upper value U=59.65 million. Conservatively assuming that this range represents 95% subjective confidence interval of the forecasters, the new 95% lower limit (LN) and upper limit (UN) can be obtained as LN=R-4.1·(R-L)= 55.20 million, UN=R+4.1·(U-R)=62.11 million. The factor 4.1 can be read from Figure 2; it is the ratio of x values corresponding to cumulative probability value S=0.05 for u=0 ($x_{u=0}$=1.96) and u=3 ($x_{u=3}$=8.2). Incidentally, for 1972 projections for the French population in 1985 x=-5.3 so that large deviations from the previous projections for this country already occurred.

5 Sea-level Rise

Estimating changes in global sea-level due to greenhouse warming is a natural application of this technique of uncertainty characterization. The causal sequence leading to sea-level rise is as following: population -> energy production -> CO_2 emissions -> greenhouse warming -> sea-level rise. Roughly speaking, one can present the sea-level rise as a product of five factors:

$$h = Population \cdot \left(\frac{energy}{person}\right) \cdot \left(\frac{CO_2}{energy}\right) \cdot \left(\frac{\Delta T}{CO_2}\right) \cdot \left(\frac{h}{\Delta T}\right) \tag{7}$$

The first factor is the world population; the second factor is energy production *per capita*; the third factor is CO_2 emissions per unit energy production; the fourth factor is temperature increase ΔT *per* unit rise in CO_2; the fifth factor is sea-level rise *per* unit temperature increase ΔT. For each factor there are uncertainties in the respective models. In particular the last two factors include uncertainties in physical models of climate system and sea-level that our analysis suggests should be prudently described by u~1. We shall illustrate the effect of expanded confidence intervals by application of our method to the results of Oerlemans [11] who assumed Gaussian uncertainties in the physical model for sea-level rise. He used a simple fit to temperature perturbation based on a "Business-as-Usual" [12] scenario for the emission of greenhouse gases: $T = \alpha(t - 1850)^3$, where t is time, $\alpha = 27 \times 10^{-8}$ °K yr^3 and Δ is 35% of the mean. The uncertainty in individual contributions to changes in sea-level are characterized by independent normal probability distributions, hence: $\Delta^2 = \Delta^2_{glac} + \Delta^2_{ant} + \Delta^2_{green} + \Delta^2_{wais} + \Delta^2_{expa}$ + internal variability. The subscripts refer to the effect of glaciers, the Antarctic, Greenland and West Antarctic ice sheets and thermal expansion of sea water. Let us assume that the unsuspected errors in his model (which describes the complex properties and interactions of ice sheets and ocean water) are no smaller than those in measurements from nuclear and particle physics. This ignores uncertainties in other factors and therefore provides only a lower estimate of the true uncertainty.

Oerlemans projects sea-level rises with errors comparable to the estimates themselves: 33 ± 32 cm in 2050 and 65 ± 57 cm in 2100 [11]. Extreme sea-level rise, of perhaps 150 cm in 50 years, is of prime regulatory concern. A comparison of Gaussian and exponential threshold probabilities for sea-level rise by 2050 and 2100 A.D. is presented in Figure 4. We find that the Oerlemans' Gaussian model significantly underestimates the true probability distribution.

6 Summary

Fundamental physical constants appearing in the laws of nature are generally considered to be the most reliably known parameters, yet analysis of the history of measurement error and bias indicates widespread overconfidence in the accuracy of our knowledge. Similar effects are even more striking in energy and population projections. The approach taken here, that of estimating the parameter u, provides a "quick and dirty" method to quantify the uncertainties in scientific models. Our findings suggest that the parametric uncertainty of current models could be quantified by analyzing the record of prior projections and estimating the value of u. From the Figures 1 and 3 we see that u ~ 1 for physical constants and u ~ 3 for current models of population growth. It is the goal of this paper to encourage other researchers to quantify the predictive capabilities of their models by utilizing the historical trends in parameter values from previous studies.

Acknowledgements

This research was partially supported by the U.S. Department of Energy through the Northeast Center of the National Institute for Global Environmental Change (NIGEC) and by the Biological Effects Branch of the U.S. Department of Air Force through the contract F33615-92-C-0602. We thank A. Solow, M. Stoto and R. Wilson for discussion and comments; I. Shlyakhter for computer assistance. Electronic mail addresses: shlyakhter@physics.harvard.edu.; kammen@hrl.harvard.edu.

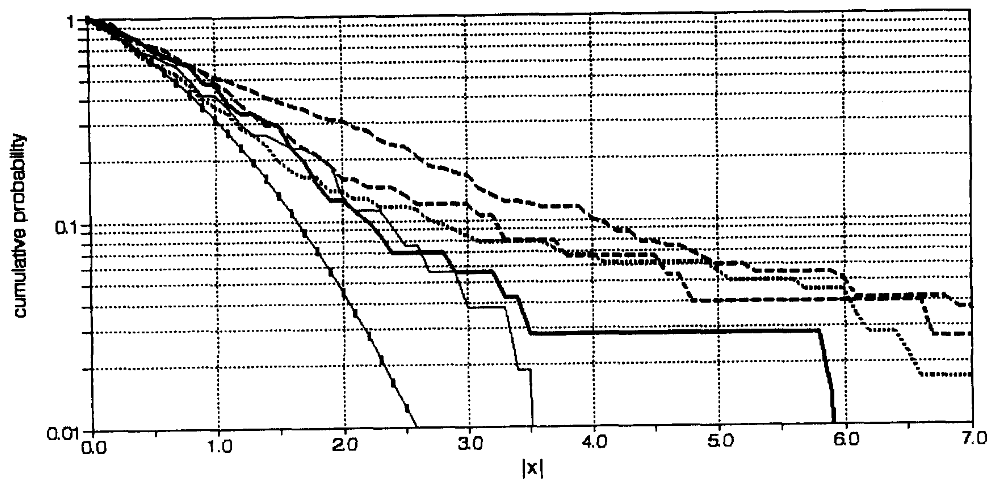

Figure 1: Probability of unexpected results in physical measurements. The plots depict the cumulative probability, $S(x)=\int_x^\infty p(t)dt$, that new measurements, *a*, will be at least $|x|$ standard deviations, Δ, away from the old results, A; $|x| = (a - A)/\Delta$ as defined in the text. The cumulative probability distributions of |x| are shown for the five data sets: elementary particles data (heavy solid line); magnetic moments and lifetimes of excited nuclear states (respectively heavy centered line and heavy dotted line), neutron scattering lengths (heavy dashed line), and average neutron resonance parameters (solid line). Also shown is the Gaussian cumulative curve (light solid line with markers). See [5] for further details.

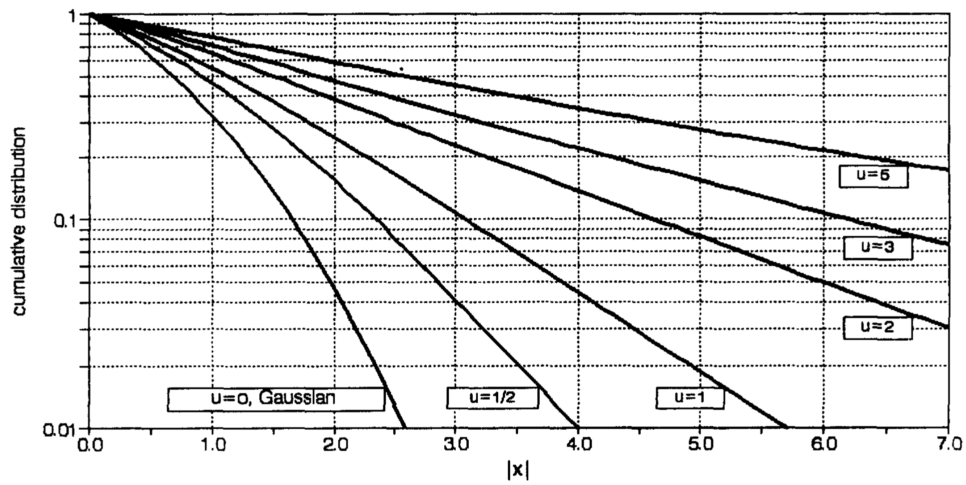

Figure 2: One-parameter family of probability distributions. *u* quantifies the uncertainty in the standard deviation of the Gaussian distribution (Eqs. (4,5)). The values of *u* are indicated in the figure. The curves demonstrate the continuum of probability distributions: from Gaussian (*u*=0) to exponential (*u* > 1).

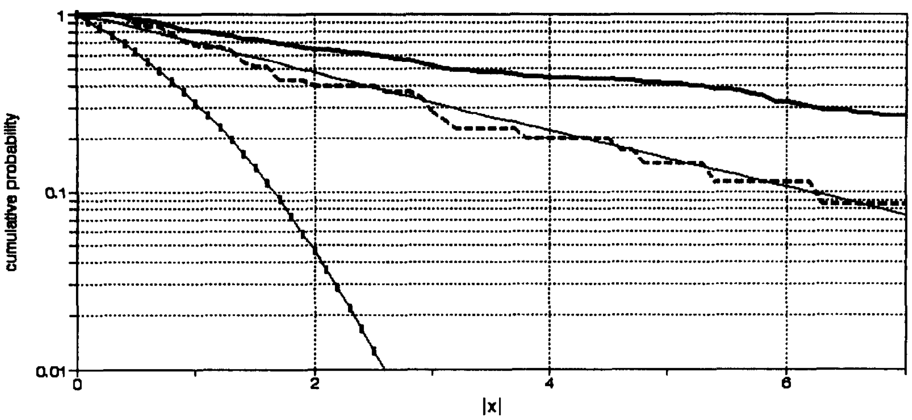

Figure 3: Population projections. The plots depict the cumulative probability, $S(x)=\int_x^{\infty} p(t)dt$, that true values (T) will be at least |x| standard deviations (Δ) away from the reference value of old projections (R). The population data base is described in the text. The cumulative probability distributions of |x| are shown for the total dataset of 133 countries (heavy solid line) and for a subset of 37 industrialized countries (heavy dashed line). Also shown is the Gaussian curve (light solid line with markers) and the curve u=3 from Figure 2 (light solid line).

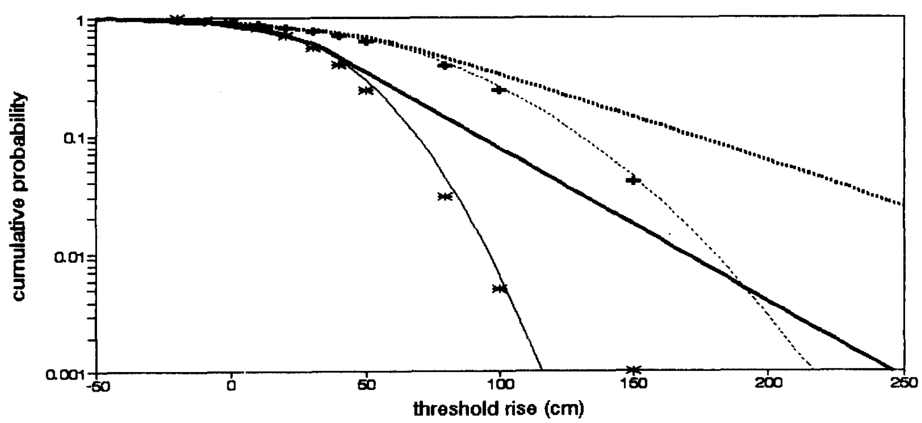

Figure 4: Projections of sea-level rise for 2050 A.D. and 2100 A.D. The probability of a sea-level rise greater than a given threshold are plotted for the normal probability (2050: thin solid line; 2100: thin dashed line) and for the exponential distribution $e^{-|x|/1.3}$ which approximates the curve *u*=1 in Figure 2. (2050: heavy solid line; 2100: heavy dashed line). Our fit to Oerlemans' calculation for 2050 A.D. results in sea-level rise of 33 ± 26 cm, and for 2100 66 ± 48 cm. The uncertainty in parameter values does not preclude a fall in sea-level (a negative sea-level rise).

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